

Distinction of the Steinberg representation

P. Broussous

With an appendix by F. Courtès

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Abstract

We prove Dipendra Prasad's conjecture on distinction of the Steinberg representation [Pr] for symmetric spaces of the form $\mathbb{G}(E)/\mathbb{G}(F)$, when \mathbb{G} is a split reductive group defined over F , and E/F an unramified quadratic extension of non-archimedean local fields.

Introduction

Let \mathbb{G} be a connected reductive group defined over a non-archimedean local field F , and let E/F be a quadratic galois extension of F . If π is a smooth representation of $\mathbb{G}(E)$ and χ a smooth character of $\mathbb{G}(F)$, one says that π is χ -distinguished if the intertwining space

$$\mathrm{Hom}_{\mathbb{G}(F)}(\pi, \chi)$$

is non-trivial.

Let \mathbf{St}_E denote the Steinberg representation of $\mathbb{G}(E)$. In [Pr], Dipendra Prasad defines an explicit quadratic abelian character χ_F of $\mathbb{G}(F)$ and makes the following conjecture.

Conjecture. ([Pr], Conjecture 3, page 77). Assume that the derived subgroup of \mathbb{G} is quasi-split. Then:

- (a) *The Steinberg representation of $\mathbb{G}(E)$ is χ_F -distinguished.*
- (b) *For any other smooth character χ of $\mathbb{G}(F)$, different from χ_F , the Steinberg representation of $\mathbb{G}(E)$ is not χ -distinguished.*

This conjecture is proved for $\mathrm{GL}(n)$ (by Prasad [Pr2] when $n = 2$, and by Anandavardhan and Rajan [AR], Theorem 1.5, for any n and without restriction on the quadratic field extension E/F).

In this article, we first prove the following result.

Theorem 1. *Assume that*

- (i) E/F is unramified,
- (ii) the residue field k_F of F is large enough.
- (iii) the algebraic group \mathbb{G} is split over F , and to make our proof less technical:
- (iv) The root system of \mathbb{G} relative to any maximal split torus is irreducible.

Then there exists an explicit quadratic character ϵ_F of $\mathbb{G}(F)$, such that \mathbf{St}_E is ϵ_F -distinguished.

We think that conditions (ii) and (iv) are not necessary. On the other hand, conditions (i) and (iii) are crucial for our proof.

Hence, in a particular case, we obtain a proof of part (a) of Prasad's conjecture modulo the fact that $\epsilon_F = \chi_F$. This equality is true for $\mathrm{GL}(n)$ and when \mathbb{G} is simply connected (in this case $\chi_F = \epsilon_F = 1$). We expect it to be always true.

The idea of the proof is to use the model of the Steinberg representation as a space of harmonic functions on the chambers of X_E , the building of $\mathbb{G}(E)$. The building X_F of $\mathbb{G}(F)$ embeds in X_E as a sub-simplicial complex of same dimension. The $\mathbb{G}(F)$ -equivariant linear form is then simply the "period" obtained by summing a fonction over the sub-building. The Iwahori-spherical vector is a test vector of this linear form. The difficult point is to prove that the restriction of a harmonic function to X_F is L^1 .

We then prove the following.

Theorem 2. *Assume that assumptions (i), (iii) and (iv) of Theorem 1 hold. Let $\mathbb{G}^{\mathrm{der}}$ be the derived group of \mathbb{G} . We have the multiplicity 1 result:*

$$\mathrm{Dim}_{\mathbb{C}} \mathrm{Hom}_{\mathbb{G}^{\mathrm{der}}(F)}(\mathbf{St}_E, \mathbb{C}) \leq 1 .$$

As a consequence, under the assumptions of Theorem 1, points (a) and (b) of Prasad's conjecture hold.

Theorem 2 is a consequence of a transitivity property of the action of $\mathbb{G}(F)$ on the chambers of X_E . The proof of this property is provided by F. Courtès in an appendix to this article.

Since the Steinberg representation factors through $\mathbb{G}(E)/Z_E$, where Z_E is the center of $\mathbb{G}(E)$, we will assume that the group \mathbb{G} is semi-simple.

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1 Notation: groups and buildings

We fix a locally compact non-archimedean and non-discrete field F . We do not assume that the (residue) characteristic of F is not 2. We let E/F be an unramified quadratic extension of F .

If K is any locally compact non-archimedean and non-discrete field, we denote by

- \mathfrak{o}_K the ring of integers of K ,
- \mathfrak{p}_K the maximal ideal of \mathfrak{o}_K ,
- $k_K = \mathfrak{o}_K/\mathfrak{p}_K$ the residue field,
- $q_K = |k_K|$ the cardinal of k_K .

We in particular have $q_E = q_F^2$.

We fix a connected semisimple group \mathbb{G} split and defined over F . We denote by d its rank and by $G = G_F$ its group of F -rational points. For simplicity, We shall assume that the root system of \mathbb{G} is irreducible.

We fix a maximal split torus \mathbb{T} of \mathbb{G} and we denote by N the normalizer of $\mathbb{T}(F)$ in G . Let T^0 be the subgroup of T generated by the $\xi(u)$, where ξ runs over the rational cocharacters of \mathbb{T} and u over \mathfrak{o}_F^\times . Then T^0 is the maximal compact subgroup of T .

Let $X = X_F$ be the semi-simple Bruhat-Tits building of \mathbb{G} . This is a locally compact topological space on which G acts continuously. It has dimension d . The space X is naturally the geometric realization of a simplicial complex and the group G acts by preserving the simplicial structure.

Let us fix a chamber C_0 in the apartment A of X attached to T and write I for the Iwahori subgroup of G attached to C_0 . This is the pointwise stabilizer of C in G .

By [IM] (also see [I]), the affine Weyl group $W = N(T)/T^0$ of T may be written as a semidirect product $W = \Omega \ltimes W_0$ of a coxeter group W_0 by a finite abelian group Ω , in such a way that:

- (a) Ω normalizes I ,

(b) if N^0 is the inverse image of W_0 in N , (I, N_0) is a Tits system (or BN -pair),

(c) the set

$$G_0 = IW_0I = \bigcup_{w \in W_0} IwI$$

is a normal subgroup of G and $G/G_0 \simeq \Omega$.

The pair (I, N) is a *generalized Tits system*. When \mathbb{G} is simply connected, we have $\Omega = \{1\}$.

As a simplicial complex X_F is the building of the BN -pair (I, N_0) [BT]. In particular X_F is labellable (in the sense of [Br], Appendix C, page 29) and G_0 acts on X_F by preserving the labelling of simplices.

The group Ω acts on A_0 and stabilizes the chamber C_0 . For $\omega \in \Omega$, we denote by $\epsilon(\omega)$ the signature of the permutation induced by the action of ω on the vertex set of C_0 . We define a quadratic character ϵ_{G_F} of G_F by

$$\epsilon_{G_F} = \epsilon \circ p_0$$

where $p_0 : G \longrightarrow G/G_0$ denotes the canonical projection.

We fix an unramified quadratic extension E/F . By [T] there is a canonical embedding

$$j : X_F \longrightarrow X_E$$

of X_F in the semisimple building X_E of \mathbb{G} over E . The Galois group $\text{Gal}(E/F)$ acts on X and j is $\text{Gal}(E/F) \ltimes G_E$ -equivariant. Moreover since \mathbb{G} is split and E/F is unramified, we have that :

- $j(X_F)$ is the set of $\text{Gal}(E/F)$ -fixed points in X_E ,
- j is simplicial.
- X_F and X_E share the same dimension d , and j maps chambers to chambers.

We shall identify X_F as a subsimplicial complex of X_E by viewing j as an inclusion.

Let Δ_d be the standard abstract simplex of dimension d . We view its set of simplices as the power set of $\{0, 1, \dots, d\}$. Since X_E is labellable, there exists a simplicial map

$$\lambda_E : X_E \longrightarrow \Delta_d$$

which preserves the dimension of simplices. If σ is a simplex of X_E , we call $\lambda_E(\sigma)$ its *label* or *type*. The restriction $\lambda_F = (\lambda_E)|_{X_F}$ is a labelling of X_F preserved by the action of G_0 .

Let C be a chamber of C and $g \in G_E$. Let (s_0, \dots, s_d) (resp. (t_0, \dots, t_d)) be an ordering of the vertices of C (resp. of gC) such that $\lambda_E(s_i) = \{i\}$, $i = 0, \dots, d$ (resp. $\lambda_E(t_i) = i$, $i = 0, \dots, d$). We denote by $\epsilon(g, C)$ the signature of the permutation:

$$\begin{pmatrix} g.s_0 & g.s_1 & \dots & g.s_d \\ t_0 & t_1 & \dots & t_d \end{pmatrix}$$

Lemma 1.1 *With the previous notation, we have:*

- (i) *the signature $\epsilon(g, C)$ does not depend on C .*
- (ii) *The map $g \mapsto \epsilon(g) = \epsilon(g, C_0)$ is a character of G_E .*
- (iv) *The character ϵ satisfies $\epsilon|_{G_F} = \epsilon_{G_F}$.*

Proof. It is easy and based on the fact that the subgroup G_E^* of G_E , formed of those elements preserving the labelling λ_E , acts transitively on chambers of X_E . Details are left to the reader. \square

2 The Steinberg representation

There are several equivalent definitions of the Steinberg representation \mathbf{St}_E of G_E . That we shall use comes from the following beautiful theorem due to Borel and Serre.

Theorem 2.1 (BS) *The representation of G_E in $H_c^d(X_E, \mathbb{C})$, the d -th cohomology space with compact support, with coefficient in \mathbb{C} , where d is the E -rank of \mathbb{G} , is equivalent to the Steinberg representation.*

Let Ch_E denote the set of chambers of X_E and $\mathbb{C}[\text{Ch}_E]$ the \mathbb{C} -vector space of complex valued functions on Ch_E of arbitrary support. A function $f \in \mathbb{C}[\text{Ch}_E]$ is called a *harmonic cocycle* if for all codimension 1 simplex D of X_E , we have

$$\sum_{C \supset D} f(C) = 0$$

where the sum is over the chambers of X_E that contain D as a subsimplex. We denote by $\mathcal{H}(X_E)$ the \mathbb{C} -vector space of harmonic cocycles on X_E .

We define a linear representation $(\pi_E, \mathcal{H}(X_E))$ of G_E in $\mathcal{H}(X_E)$ by the formula:

$$[\pi_E(g).f](C) = \epsilon(g)f(g^{-1}C), \quad g \in G, \quad C \in \text{Ch}_E.$$

This representation is not smooth in general and we denote by $(\pi_E, \mathcal{H}(X_E)^\infty)$ its smooth part.

Proposition 2.2 *The representation $(\pi_E, \mathcal{H}(X_E)^\infty)$ is equivalent as a G_E -representation to the contragredient of \mathbf{St}_E .*

Proof. For $k = d - 1, d$, let $C_c^k(X_E)^{\text{alt}}$ be the \mathbb{C} -vector space of alterned k -cochains on X_E with coefficients in \mathbb{C} , the field of complex numbers. Denote by Ch_E^* the set of pairs (C, σ) formed of a chamber C of X_E together with a bijection σ from the vertex set of C to $\{0, 1, \dots, d\}$. We let G_E act on Ch_E^* by $g.(C, \sigma) = (g.C, \sigma \circ g^*)$, where g^* is the bijection from the vertex set of C to the vertex set of $g.C$ induced by g . Then $C_c^d(X_E)^{\text{alt}}$ is the set of maps $f : \text{Ch}_E^* \longrightarrow \mathbb{C}$ satisfying:

- f has finite support,
- for all $(C, \sigma) \in \text{Ch}_E^*$ and for all permutation τ of $\{0, \dots, d - 1\}$, we have

$$f(C, \tau \circ \sigma) = \epsilon(\tau)f(C, \sigma)$$

where $\epsilon(\sigma)$ denotes the signature of σ .

The group G_E naturally acts on $C_c^d(X_E)^{\text{alt}}$. Similarly we define the G_E -module $C_c^{d-1}(X_E)^{\text{alt}}$. The coboundary map

$$d : C_c^{d-1}(X_E)^{\text{alt}} \longrightarrow C_c^d(X_E)^{\text{alt}}$$

is given by

$$dh(C, \sigma) = \sum_{D \subset C} h(D, \sigma|_D), \quad (C, \sigma) \in \text{Ch}_E^*$$

where $\sigma|_D$ denotes the restriction of σ to the vertex set of D .

For $k = d - 1, d$, let $C_c^k(X_E)$ be the \mathbb{C} -vector space of usual k -cochains with finite support. By orienting the simplices of X_E thanks to the labelling λ , we obtain a coboundary map $d : C_c^{d-1}(X_E) \longrightarrow C_c^d(X_E)$ given by

$$dh(C) = \sum_{D \subset C} (-1)^{\lambda(C \setminus D)} h(D).$$

For $k = d - 1, d$, we have an isomorphism of \mathbb{C} -vector spaces

$$C_c^k(X_E)^{\text{alt}} \longrightarrow C_c^k(X_E)$$

given by

$$f \mapsto \{C \mapsto f(C, \lambda|_C)\}$$

where $\lambda|_C$ denotes the restriction of the labelling λ to the vertex set of the simplex C . These isomorphisms are G_E -equivariant if one lets G_E act on $C_c^k(X_E)$ via

$$[g.f](D) = \epsilon_{G_E}(g) f(g^{-1}.D), \quad D \text{ } k\text{-simplex of } X_E.$$

Moreover the isomorphisms are compatible with the coboundary maps.

The space $H_c^d(X_E)$ is known to be isomorphic to $C_c^d(X_E)^{\text{alt}}/dC_c^{d-1}(X_E)^{\text{alt}}$ as a G_E -module. So it is isomorphic to $C_c^d(X_E)/dC_c^{d-1}(X_E)$ as a G_E -module.

By letting V^* denote the algebraic dual of a \mathbb{C} -vector space V , we have

$$(H_c^d(X_E))^* = \{\omega \in C_c^d(X_E)^* ; f_{|dC_c^{d-1}(X_E)} = 0\} .$$

We may identify $C_c^d(X_E)^*$ with $\mathbb{C}[\text{Ch}_E]$ by using the pairing

$$\langle \omega, f \rangle = \sum_{C \in \text{Ch}_E} \omega(C) f(C) , \quad \omega \in \mathbb{C}[\text{Ch}_E], \quad f \in C_c^d(X_E) .$$

Then for $\omega \in \mathbb{C}[\text{Ch}_E]$, the condition $\omega_{|dC_c^{d-1}(X_E)} = 0$ writes $\langle \omega, dh \rangle = 0$, for all $h \in C_c^{d-1}(X_E)$. This may be rewritten

$$\langle d^* \omega, h \rangle = 0 , \quad h \in C_c^{d-1}(X_E) \text{ that is } d^* \omega = 0$$

where $d^* : C_c^d(X_E)^* \longrightarrow C_c^{d-1}(X_E)^*$ is the adjoint of d . But a simple computation shows that

$$d^* \omega(D) = \sum_{C \supset D} \omega(C) , \quad D \text{ (} d-1 \text{)-simplex}$$

so that $d^* \omega = 0$ is the harmonicity condition. \square

Note that the Steinberg representation of G_E is self-dual.

3 Some geometric lemmas

We denote by d_g the combinatorial distance on X_E defined as follows. For $C, D \in \text{Ch}_E$, $d_g(C, D)$ is the length k of a minimal gallery (D_0, D_1, \dots, D_k) satisfying $D_0 = C$ and $D_k = D$. The following result, due to F. Bruhat, will be very useful.

Lemma 3.1 (*Lemma 4.1 of [Bo]*) *Let U be a compact open subgroup of G_E . There exists an integer $k_0 = k_0(U)$ satisfying the following property. For all chamber C such that $d_g(C_0, C) \geq k_0$, there exists a chamber D adjacent to C such that:*

- (i) $d_g(C_0, D) = d(C_0, C) - 1$;
- (ii) *the group U acts transitively on the set of chambers C' such that $C' \neq D$ and $C' \cap D = C \cap D$.*

Lemma 3.2 *Let D be a codimension 1 simplex in X_F (resp. in X_E). Then D is contained in $q_F + 1$ chambers of X_F (resp. in $q_E + 1$ chambers of X_E).*

Proof. We give a proof for X_F . Let P_D be the parahoric subgroup of G_F attached to D and P_D^1 its pro-unipotent radical. Then $P_D/P_D^1 = \mathbb{G}_D(k_F)$, where \mathbb{G}_D is a reductive group defined over k_F and of k_F -rank 1. The chambers C of X_F containing D are in bijection with the Borel subgroup of P_D/P_D^1 by

$$C \mapsto P_C \bmod P_D^1$$

where P_C denotes the Iwahori subgroup attached to C . But \mathbb{G}_D being of k_F -rank 1, $\mathbb{G}_D(k_F)$ possesses $q_F + 1$ Borel subgroups. \square

For any non negative integer k , we denote by $\Sigma_F(k)$ the set of chambers of X_F at distance k from C_0 and set $N_k(k) = |\Sigma_F(k)|$.

Lemma 3.3 *We have*

$$N_F(k) \leq (d+1) d^{k-1} q_F^k, \quad k \geq 1.$$

Proof. Any chamber of X_F has $d+1$ codimension 1 faces. A chamber in $\Sigma_F(1)$ contains one of the $d+1$ codimension 1 faces of C_0 . By Lemma (3.2), such a face is contained in q_F chambers different from C_0 , so that

$$N_F(1) = (d+1) q_F.$$

Moreover, for $k \geq 1$, any chamber in $\Sigma_F(k)$ is adjacent to at most dq_K chambers at distance from C_0 greater than k . The formula follows by induction on k . \square

Lemma 3.4 *Let $f \in \mathcal{H}(X_E)^\infty$. There exist an integer k_f and a positive real number K_f such that the following holds. For all $C \in \text{Ch}_E$ such that $d_g(C_0, C) \geq k_f$, we have*

$$|f(C)| \leq K_f \cdot q_E^{-d_g(C_0, C)}.$$

Proof. Since f is smooth under the action of G , it is fixed by an open compact subgroup U small enough. Set $k_f = k_0(U)$. For $k \geq 0$, set

$$M_k = \text{Max} \{ |f(C)| ; C \in \Sigma_E(k) \}.$$

We are going to prove that for $k \geq k_f$ we have $M_{k+1} \leq q_E^{-1} M_k$; the lemma will follow.

Let $C \in \Sigma_E(k+1)$. By applying Lemma (3.1), there exists $D \in \Sigma_E(k)$ such that U acts transitively on

$$[C, D] := \{G \in \text{Ch}_E ; G \neq D \text{ and } G \cap D = C \cap D\} .$$

It follows that f is constant on $[C, D]$. By applying the harmonicity condition at the codimension 1 face $C \cap D$, we get

$$q_E f(C) + f(D) = 0 ,$$

since $[C, D]$ has q_E elements. So $|f(C)| = q_E^{-1} |f(D)|$, and our assertion follows. \square

Lemma 3.5 *Assume that $q_F > d$. Let $f \in \mathcal{H}(X_E)^\infty$. Then we have*

$$f|_{\text{Ch}_F} \in L^1(\text{Ch}_F)$$

where $L^1(\text{Ch}_F)$ denotes the set of complex functions g on Ch_F such that

$$\sum_{C \in \text{Ch}_F} |g(C)| < +\infty .$$

Proof. We may write

$$\sum_{C \in \text{Ch}_F} |f(C)| = \sum_{k \geq 0} \sum_{C \in \Sigma_F(k)} |f(C)| .$$

By the previous lemmas, for k large enough and for some constant $K > 0$, we have:

$$\sum_{C \in \Sigma_F(k)} |f(C)| \leq K \left(\frac{dq_F}{q_E} \right)^k .$$

with $q_E = q_F^2$. The result follows. \square

Remark. If \mathbb{G} is of rank 1, then the condition $q_F > d$ is automatically satisfied.

4 Constructing G_F -equivariant linear forms

In this section, we assume, as in Lemma (3.5), that we have $q_F > d$.

Thanks to lemma (3.5), the linear map λ on $\mathcal{H}(G_E)^\infty$ given by

$$\lambda(f) = \sum_{C \in \text{Ch}_F} f(C)$$

is well defined. For $g \in G_F$ and $f \in \mathcal{H}(G_E)^\infty$, we have

$$\lambda(\pi_E(g).f) = \sum_{C \in \text{Ch}_F} \epsilon_{G_F}(g) f(g^{-1}C) = \epsilon_{G_F}(g) \lambda(f) .$$

Hence we have $\lambda \in \text{Hom}_{G_E}(\mathbf{St}_E, \epsilon_{G_F})$.

Theorem 4.1 *The Steinberg representation of G_E is ϵ_{G_F} -distinguished. More precisely, a non-zero Iwahori-spherical vector is a test vector for λ .*

Proof. It suffices to prove that λ is not trivial. Let f be the Iwahori-spherical vector in $\mathcal{H}(G_E)^\infty$ normalized in such a way that $f(C_0) = 1$. In Lemma (3.1), if $U = I$, we may take $k_0 = 0$. It follows from the proof of Lemma (3.4) that, for all $k \geq 0$, f has constant value $(\frac{-1}{q_E})^{-k}$ on $\Sigma_F(k)$. As a consequence

$$\lambda(f) = \sum_{k \geq 0} \left(\sum_{C \in \Sigma_F(k)} f(C) \right)$$

is an alternating series. In particular we have

$$\sum_{C \in \Sigma_F(0)} f(C) > \lambda(f) > \sum_{C \in \Sigma_F(0)} f(C) + \sum_{C \in \Sigma_F(1)} f(C)$$

that is

$$1 > \lambda(f) > 1 - \frac{d+1}{q_F} \geq 0 .$$

and our Theorem follows. \square

Note that if the F -rank of \mathbb{G} is 1 the value $\lambda(f)$ may be explicitly computed. Indeed in that case, X_F is a regular tree of valence $q_F + 1$, and we have $N_F(k) = 2q_F^k$, $k \geq 1$. Hence

$$\lambda(f) = 1 + \sum_{k \geq 1} 2q_F^k \left(\frac{-1}{q_E} \right)^k = 1 - \frac{2}{q_F + 1} .$$

5 Multiplicity 1

In this section we release the condition $q_F > d$ and prove Theorem 2 without restriction on the size of k_F .

Set $\mathbb{H} = \mathbb{G}^{\text{der}}$ and $H = \mathbb{H}(F)$. Note that \mathbb{H} and \mathbb{G} share the same (semisimple) Bruhat-Tits building over F (resp. over E). This essentially

comes from the fact that the inclusion $H \longrightarrow G$ is $B - N$ -adapté in the sense of [BT] (1.2.13), page 18 (cf. [BT] §2.7., page 49). By Proposition (2.2), we have

$$\mathrm{Hom}_H(\mathbf{St}_E, \mathbb{C}) = \mathcal{H}(X_E)^H ,$$

the space of harmonic cochains on X_E fixed by H .

Our proof relies on the following fundamental result whose proof is given in the appendix.

Theorem 5.1 *Let C be a chamber of X_E at combinatorial distance $\delta \geq 0$ from X_F . Then G_F acts transitively on the set $\mathrm{Ch}[C, \delta + 1]$ of chambers D of X_E satisfying:*

- D and C are adjacent,
- $d(D, X_F) = \delta + 1$.

Note that since H is contained in G_0 (G/G_0 is abelian), it acts on $\mathcal{H}(X_E)$ via the formula

$$h.\omega(C) = \omega(h^{-1}C) , \quad h \in H, \quad \omega \in \mathcal{H}(X_E) .$$

Let $\omega \in \mathcal{H}(X_E)^H$. Since H acts transitively on X_F , the value $\omega(C)$ does not depend on the chamber C of X_F . Let us denote it by $\varphi(\omega)$. Theorem 2 is a consequence of the following:

Lemma 5.2 *The linear map*

$$\varphi : \mathcal{H}(X_E)^H \longrightarrow \mathbb{C} , \quad \omega \mapsto \varphi(\omega)$$

is injective.

Proof. For all integers $\delta \geq 0$, let $\mathrm{Ch}(X_F, \delta)$ denote the set of chambers in X_E at combinatorial distance δ from X_F (in particular $\mathrm{Ch}(X_F, 0)$ is the set of chambers of X_F). Let $\omega \in \mathcal{H}(X_E)^H$ and $\delta \geq 0$ be an integer. We prove that the restriction $\omega|_{\mathrm{Ch}(X_F, \delta+1)}$ is entirely determined by the restriction $\omega|_{\mathrm{Ch}(X_F, \delta)}$. The lemma will obviously follow.

Let $D \in \mathrm{Ch}(X_F, \delta + 1)$. Fix a chamber $C \in \mathrm{Ch}(X_F, \delta)$ adjacent to D and set $M = C \cap D$. The harmonicity condition at the codimension 1 face M writes

$$\sum_{\Delta \in C_M} \omega(\Delta) = 0 ,$$

where C_M is the set of chambers of X_E containing M . We may split the set C_M into two subsets : $C_M^{\delta+1} := \text{Ch}[C, \delta + 1]$ and its complement C_M^δ , contained in $\text{Ch}(X_F, \delta)$. By theorem (5.1) and the H -invariance of ω , we have

$$\sum_{\Delta \in C_M^{\delta+1}} \omega(\Delta) = |C_M^{\delta+1}| \times \omega(C) .$$

Hence the harmonicity condition gives

$$\omega(C) = -\frac{1}{|C_M^{\delta+1}|} \times \sum_{\Delta \in C_M^\delta} \omega(\Delta)$$

This proves that the value $\omega(C)$ depends only on the restriction $\omega|_{\text{Ch}(X_F, \delta)}$, and we are done. \square .

A A transitivity result

For every facet $A \subset X_E$, we shall denote by $K_{A,E}$ (resp $K_{A,F}$) the connected fixator of A in G_E (resp. the intersection with G_F of that connected fixator). More generally, for every subset S of X_E , we shall denote by $K_{S,E}$ (resp. $K_{S,F}$) the intersection of the $K_{A,E}$ (resp. $K_{A,F}$), where A runs over the set of facets of X_E whose intersection with S is nonempty. Let \overline{S} be the closure of S ; we have of course $K_{\overline{S},E} = K_{S,E}$ and $K_{\overline{S},F} = K_{S,F}$.

Proposition A.1 *Let d be a nonnegative integer, and let C be any chamber of X_E such that the combinatorial distance between C and X_F is d . Let C' be a chamber of X_E neighbouring C and whose combinatorial distance from X_F is $d + 1$, let A be the unique facet of codimension 1 of X_E contained in both \overline{C} and $\overline{C'}$, and let Δ be the set of chambers of X_E containing A in their closure and whose combinatorial distance from X_F is $d + 1$. Then the group $K_{C,F}$ acts transitively on Δ .*

Proof. Assume first $d = 0$, that is C is contained in X_F . Let \mathcal{A} be an apartment of X_F containing C , let T be the corresponding F -split maximal torus of G_E , let Φ be the root system of G_E relatively to T_E and let $\pm\alpha$ be the elements of Φ corresponding to the hyperplane H of \mathcal{A} containing A . For every $\beta \in \Phi$, let $U_\beta = U_{\beta,E}$ be the corresponding root subgroup of G_E , and let v be a normalized valuation (that is a valuation such that for every β , $v(U_\beta) = \mathbb{Z} \cup \{\infty\}$; such a valuation exists because \mathbb{G} is split over an unramified extension of E) on the root datum $(G, T, (U_\beta))$ such that, with the subgroups $U_{\beta,i}$ of U_β being defined according to that valuation, we have $U_{\pm\alpha} \cap K_A = U_{\pm\alpha,0}$; we shall assume α is the one such that $U_\alpha \cap K_C = U_{\alpha,1}$.

Let ϕ (resp. ϕ') be a F -isomorphism between E and $U_{\alpha,E}$ (resp. $U_{-\alpha,E}$) preserving the valuation, that is such that for every integer i , $U_{\alpha,i}$ (resp. $U_{-\alpha,i}$) is the image by ϕ (resp. ϕ') of the elements of E of valuation $\geq i$; the elements of Δ are the chambers of the form $\phi(x)C$, where x is an element of \mathfrak{o}_E which belongs neither to F nor to \mathfrak{p}_E ; moreover, $\phi(x)C$ depends only of the class of x modulo \mathfrak{p}_E . we can thus label the elements of Δ as $C_x = \phi(x)C$, where x is an element of $k_E - k_F$.

Let now Φ^\vee be the system of coroots of T_E associated to Φ , and let α^\vee be the 1-parameter subgroup of T_E corresponding to α in Φ^\vee ; for every $y \in \mathfrak{o}_E^*$, we have $\alpha^\vee(y)C = C$, and if y is an element of $\mathfrak{o}_F^* + \mathfrak{p}_E$, $\alpha^\vee(y)$ permutes the elements of Δ . Moreover, $\alpha^\vee(y)$ depends only of the class of y modulo $1 + \mathfrak{p}_E$, hence we can view y as an element of k_E^* .

Let x be an element of $k_E - k_F$ (arbitrarily fixed for the moment). For every $a \in k_F^*$ and every $b \in k_F$, we have:

$$\phi(b)\alpha^\vee(a)C_x = \phi(b)(Ad(\alpha^\vee(a))\phi(x))C = \phi(b)\phi(a^2x)C = C_{a^2x+b}.$$

Hence for every element of $k_E - k_F$ of the form $y = a^2x + b$, C_y is in the G_F -orbit of C_x . If $\text{char}(k_E) = 2$, every element of k_F is a square, and since $(1, x)$ is a basis of the k_F -vector space k_E , every element of $k_E - k_F$ is of that form, which proves the proposition in that case.

Now assume $p \neq 2$; there exists then $x \in k_E - k_F$ such that $\frac{1}{c} = x^2 \in k_F$; c is then not a square in k_F . Let D be the chamber of \mathcal{A} such that $\overline{D} \cap \overline{C} = \overline{A}$; we have $D = nC$, where n is any representative in the normalizer of K_T in K_A of the element s_α in the Weyl group of Φ . Moreover, according to [BT, 6.1.3 a) and b)], we can assume that every such element is of the form:

$$n = \phi'(y)\phi(-y^{-1})\phi'(y),$$

with $y \in \mathfrak{o}_E^*$. We then have:

$$\phi'(y)D = \phi'(y)\phi'(-y)\phi(y^{-1})\phi'(-y)C = C_{y^{-1}},$$

since $\phi'(y)C = C$. Hence $C_x = \phi(x^{-1})D = \phi(xc)D$. By the same reasoning as above, for every $a \in k_F^*$ and every $b \in K_F$, $\phi'(a^2xc + b)D = C_{\frac{1}{a^2xc+b}}$. On the other hand, we have:

$$\frac{1}{a^2xc + b} = \frac{a^2xc - b}{(a^2xc + b)(a^2xc - b)} = \frac{a^2xc - b}{a^4c - b^2} = \frac{x - \frac{b}{a^2c}}{a^2 - \frac{b^2}{a^2c}}.$$

On the other hand, it is well-known and easy to check that there exists a, b such that $a^2 - \frac{b^2}{a^2c}$ is not a square; we thus obtain that there exists a', b' , such

that a' is not a square and $C_{a'x+b'}$ is in the $K_{C,F}$ -orbit of C_x . By the same reasoning as above once again, we obtain that it is true for every $C_{a'a^2x+b+b'}$, $a \in k_F^*$, $b \in k_F$. Since $(k_F^*)^2$ is of index 2 in k_F , we finally obtain that all of the C_x , $x \in k_E - k_F$, are in the same $K_{C,F}$ -orbit, which completes the proof of the proposition when $C \subset X_F$.

Now assume $d > 0$. Set $\Gamma = \text{Gal}(E/F)$, and let γ be the unique nontrivial element of Γ . First we prove the following lemma:

Lemma A.2 *There exists a Γ -stable apartment of X_E containing both C and $\gamma(C)$.*

Proof. Let \mathcal{A} be any apartment of X_E containing both C and $\gamma(C)$; such an apartment exists by [BT, proposition 2.3.1]. Obviously, $\gamma(\mathcal{A})$ satisfies the same property; there exists then $g \in G_E$ such that $g\mathcal{A} = \gamma(\mathcal{A})$, and we can assume $g \in K_{C,E} \cap K_{\gamma(C),E}$. The element $\gamma(g)$ then also belongs to $K_{C,E} \cap K_{\gamma(C),E}$, and we have $\gamma(g)\gamma(\mathcal{A}) = \mathcal{A}$. Hence $\gamma(g)g$ fixes \mathcal{A} pointwise, which means that it belongs to the unique parahoric subgroup K_T of the E -split maximal torus T of G_E associated to \mathcal{A} .

Let now F_{nr} be the maximal unramified extension of F , let $G_{F_{nr}}$ be the group of F_{nr} -points of \mathbb{G} , and let $K_{C,F_{nr}}$ be the connected fixator of C viewed as a chamber of the Bruhat-Tits building $X_{F_{nr}}$ of $G_{F_{nr}}$. By [Cou, lemma 5.1], there exists an element $h \in K_{F_{nr}}$ such that $g = \mathbf{F}(h)^{-1}h$, with \mathbf{F} being the Frobenius element of $\text{Gal}(F_{nr}/F)$. Moreover, the restriction of \mathbf{F} to E is γ , and we have:

$$\gamma(g)g = \mathbf{F}^2(h)^{-1}h \in K_T,$$

Let T_{nr} be the maximal torus of $G_{F_{nr}}$ associated to \mathcal{A} , and let $K_{T_{nr}}$ be its unique parahoric subgroup; we have $K_T = K_{T_{nr}} \cap G_E$. Moreover, the Frobenius element of $\text{Gal}(F_{nr}/E)$ is \mathbf{F}^2 ; by [Cou, lemma 5.1] again, there exists then $t \in K_{T_{nr}}$ such that $\gamma(g)g = \mathbf{F}^2(t)t^{-1}$. Hence $ht = \mathbf{F}^2(ht)$, which simply means that $ht \in G_E$. We finally obtain:

$$ht\mathcal{A} = h\mathcal{A} = \gamma(h)\gamma(\mathcal{A}) = \gamma(h\mathcal{A}) = \gamma(ht\mathcal{A}),$$

hence $ht\mathcal{A}$ is a Γ -stable apartment of X_E containing both C and $\gamma(C)$ and the lemma is proved. \square .

Now we designate by \mathcal{A} the apartment given by the above lemma, and by T the corresponding E -split maximal torus of G_E ; T is defined over F , but not F -split. Let Φ be the root system of G_E relatively to T , and let $\alpha \in \Phi$ be defined as in the case $d = 0$. Since T is defined over F , Γ acts on Φ .

Let D be the unique chamber of \mathbf{A} such that $\overline{C} \cap \overline{D} = \overline{A}$. Since Δ is nonempty, the combinatorial distance between D and \mathbf{B}_F must be either d or $d + 1$.

Lemma A.3 *Assume $H = \gamma(H)$. Then the combinatorial distance between D and \mathcal{F} is d .*

Proof. Let s_H be the orthogonal reflection on \mathcal{A} whose kernel is H . Since $H = \gamma(H)$, γ and s_H commute, hence there exists $g_H \in G_F$ such that g_H acts on \mathcal{A} via s_H . Let $C = C_0, \dots, C_d$ be a minimal gallery of length d between C and some chamber C_d of X_F . Then $D = g_H C_0, \dots, g_H C_d$ is also a minimal gallery and $g_H C_d \subset X_F$, hence the combinatorial distance between D and X_F is at most d . The other inequality follows from the above remarks. \square

Note that the fact that $H = \gamma(H)$ implies in particular that $\gamma(\alpha) = \pm\alpha$. Conversely, we have:

Lemma A.4 *Assume $\gamma(\alpha) = \alpha$. Then $H = \gamma(H)$.*

Proof. Let $\bar{\alpha}$ be the affine root of T corresponding to H ; it is an affine linear form on the affine space \mathcal{A} , and the corresponding linear form on the vector space $(X_*(T)/X_*(Z)) \times \mathbb{R}$, where Z is the center of G , is α . Hence $\gamma(\bar{\alpha})$ is of the form $\bar{\alpha} + c$, with c being some constant. We then have $\gamma^2(\bar{\alpha}) = \bar{\alpha} + 2c$; since γ^2 is trivial, it implies $c = 0$, hence $H = \gamma(H)$. \square

Note that it is not true when $\gamma(\alpha) = -\alpha$.

Now we prove the proposition when $H = \gamma(H)$. Consider the rank 1 subgroup G_α of G_E generated by T , U_α and $U_{-\alpha}$; it is defined over F , and the fact that $H = \gamma(H)$ implies that $G_\alpha \cap K_A = G_\alpha \cap K_{\gamma(A)}$, hence K_A is Γ -stable. The elements of Δ are then of the form uC , where u is an element of U_α not belonging to G_F , and we can finish the proof the same way as in the case $d = 0$.

Assume now $H \neq \gamma(H)$. Let \mathcal{C} be the connected component of $\mathcal{A} - (H \cup \gamma(H))$ containing C . Assume \mathcal{C} contains $\gamma(C)$ as well. Consider an apartment of X_E of the form $\phi(x)\mathcal{A}$, where ϕ is defined as in the case $d = 0$ for a given normalized valuation v on $(G, T, (U_\beta))$, and x is an element of E of valuation i , where i is such that $U_\alpha \cap K_{C,E} = U_{\alpha,i}$. Then $\phi(x)\mathcal{A}$ contains at the same time a chamber C'' distinct from C whose closure contains A and the half-apartment of \mathcal{A} delimited by H and containing C , which itself contains the closure of $C \cup \gamma(C) \cup \gamma(D)$. We deduce from this that we have $\gamma(\phi(x))\phi(x)\gamma(D) = \gamma(C'')$, hence $\phi(x)\gamma(\phi(x))D = C''$.

Moreover, $\phi(x)\gamma(\phi(x))$ is contained in $K_{C,E}$, which is a pro-solvable group, hence if $\gamma(\alpha) = -\alpha$, the commutator $[\phi(x)^{-1}, \gamma(\phi(x))^{-1}]$ is an element of the subgroup K' of $K_{C,E}$ generated by K_T , $U_{\alpha,i+1}$ and $\gamma(U_{\alpha,i+1})$, which is itself contained in $K_{D \cup \gamma(D), E}$. If now $\gamma(\alpha) \neq \pm\alpha$ (remember that by the previous lemma we cannot have $\gamma(\alpha) = \alpha$), then $[\phi(x)^{-1}, \gamma(\phi(x))^{-1}]$ is an element of the intersection with $K_{C,E}$ of the subgroup of G generated by the $U_{\lambda\alpha + \mu\beta}$,

where λ and μ are positive integers such that $\lambda\alpha + \mu\beta$ is a root. We'll also denote by K' this last subgroup; it is also contained in $K_{D \cup \gamma(D), E}$.

In both cases, we can apply [Cou, lemma 5.1] to see that there exists $k \in K'$ such that $[\phi(x)^{-1}, \gamma(\phi(x))^{-1}] = \gamma(k)k^{-1}$, hence $\phi(x)\gamma(\phi(x))k = \gamma(\phi(x))\phi(x)\gamma(k)$. We thus have proved that $\phi(x)\gamma(\phi(x))k$ is an element of $K_{C,F}$ sending D to C'' ; since this is true for any C'' and in particular for C' , Δ must contain all of them and $K_{C,F}$ acts transitively on them, which proves the proposition in this case.

Assume now that \mathcal{C} does not contain $\gamma(C)$, or in other words that C and $\gamma(C)$ are separated by at least one of H and $\gamma(H)$. Then they are separated by both of them, which means that D and $\gamma(D)$ are in the same connected component. We can then apply the same reasoning as above with C and D switched, and we obtain that for every chamber C'' of X_E containing A in its closure and distinct from D , there exists an element g of G_F such that $gC = C''$, which implies in particular that the combinatorial distance between C'' and X_F must be d . Since by our hypothesis this is not true for C' , we must have $C' = D$ and even $\Delta = \{D\}$, and the result of the proposition is then trivial. \square

Remark. Actually, this very last case turns out to be impossible. To see that, we can for example observe that the combinatorial distance between C and X_F is equal to the combinatorial distance between C and some facet of $X_F \cap \mathcal{A}$ of maximal dimension plus the dimension of the F -anisotropic component of T , and that there exists a minimal gallery between C and some chamber of X_F whose closure contains the barycenter b of $C \cup \gamma(C)$ (which is itself an element of X_F); with the hypotheses of the last case, it is easy to check that the closure of $C \cup \{b\}$ must contain D , hence a contradiction.

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Paul Broussous

paul.broussous@math.univ-poitiers.fr

François Courtès

francois.courtes@math.univ-poitiers.fr

Département de Mathématiques

UMR 6086 du CNRS

Téléport 2 - BP 30179

Boulevard Marie et Pierre Curie

86962 Futuroscope Chasseneuil Cedex

France